Functional Quantum Theory of Scattering Processes I

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Dynamics of quantum field theory can be formulated by functional equations. To develop a complete functional quantum theory one has to describe the physical information by functional operations. One of the most important physical information of elementary particle physics is the *S*-matrix. To derive a functional expression for this quantity the potential scattering model is studied. A functional *S*-matrix is defined and its equivalence with the ordinary *S*-matrix definition in physical Hilbert space is proven. Also a calculational method for scattering functionals is proposed. In the appendices technical details are discussed.

In quantum field theory, especially in nonlinear spinor theory of elementary particles [1], the dynamical behaviour of the physical systems can be described by functionals of field operators in a Heisenberg representation and corresponding functional equations [2–6]. Using this representation of quantum dynamics it is suggesting to develop a functional quantum theory where all informations to be obtainable by conventional quantum theory can be expressed in terms of functional operations. A first attempt in this direction has been made by the introduction of an appropriate scalar product definition for physical state functionals [7–9]. This definition was shown to allow a mapping of the physical Hilbert space on to the corresponding functional Hilbert space [7, 8]. But functional quantum theory, especially for elementary particles, requires more specific information than so far has been given. One of the most important quantities of elementary particle physics defined by scalar product expressions is the *S*-matrix. Therefore in functional quantum theory of elementary particles one would like to construct a functional representation of the *S*-matrix. Naturally, to realize this program, one may not start with relativistic spinor theory itself. Rather it is very instructive for a first step in this direction to use an unproblematic completely transparent problem like potential scattering for a test investigation. In this case a rigorous proof of the equivalence of the functional and the ordinary *S*-matrix definition can be given. This is undertaken in this paper. But, to avoid lengthy deductions, not all steps of the proof are discussed in detail. Rather the investigation is confined to the essential points in order to demonstrate what really happens. So the treatment given here has to be improved further. As will be shown in a following paper, the method can be extended to nonlinear spinor theory too. The only difference between nonrelativistic and relativistic theory will turn out to be, that the latter requires more unproven although physically plausible assumptions than the former. In the following $\hbar = c = 1$ is used and for brevity the mass of the particles is assumed to be $1/2$ in the chosen system of units.

1. Interacting Matter Field

To describe potential scattering by functionals we discuss first the field-theoretic description in ordinary i. e. physical Hilbert space. Assuming a matter field operator $\psi(\mathfrak{r}, t)$, potential scattering in terms of this field operator is characterized by the Hamiltonian

$$H(t) = -\int \nabla\psi^+(\mathfrak{r}, t)\ \nabla\psi(\mathfrak{r}, t)\ d\mathfrak{r} + \int \psi^+(\mathfrak{r}, t)\ \psi^+(\mathfrak{r}', t)\ V(\mathfrak{r}-\mathfrak{r}')\ \psi(\mathfrak{r}', t)\ \psi(\mathfrak{r}, t)\ d\mathfrak{r}\ d\mathfrak{r}' \tag{1.1}$$

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[2] Y. V. Novozhilov and A. V. Tulub, The method of functionals in the quantum theory of fields, Gordon & Breach, New York 1961.
[3] W. T. Martin and I. Segal, Analysis in function space, M.I.T. Press 1963.
[4] H. Rampacher, H. Stumpf, and F. Wagner, Fortschr. Phys. **13**, 385 [1965].
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[6] E. A. Berezin, The method of second quantization, Academic Press, New York—London 1966.
[7] H. Stumpf, Z. Naturforsch. **24 a**, 188 [1969].
[8] H. Sohr, in preparation.
[9] R. Weber, Thesis, University of Tübingen 1969, in preparation.

where ψ^+ is the Hermitean conjugate of ψ and $V(r)$ is a sufficient rapidly decreasing potential. Because of the time independence of the matter field interaction $H(t)=H(0)$ for arbitrary t. The commutation relation

$$[\psi(\mathfrak{r},t)\,\psi^+(\mathfrak{r}',t)]_- = \delta(\mathfrak{r}-\mathfrak{r}')\,\mathbf{1} \tag{1.2}$$

is assumed, and all other commutators vanish. Then from (1.1) and (1.2) follows by usual procedures the field equation

$$\left(i\frac{\partial}{\partial t}+\Delta\right)\psi(\mathfrak{r},t) - \int \psi^+(\mathfrak{r}'t)\,\psi(\mathfrak{r}'t)\,V(\mathfrak{r}'-\mathfrak{r})\,d\mathfrak{r}'\,\psi(\mathfrak{r},t)=0. \tag{1.3}$$

To describe the states of this system we assume further the true ground state $|0\rangle$ of the system to be identical with the bare vacuum and

$$\psi(\mathfrak{r},t)\,|0\rangle = 0\,. \tag{1.4}$$

Then the n particle states $|a\rangle$ of the system are given by

$$|a\rangle = \int \varphi_a(\mathfrak{r}_1\ldots\mathfrak{r}_n)\,(n!)^{-\frac{1}{2}} \times \psi^+(\mathfrak{r}_1 0)\ldots\psi^+(\mathfrak{r}_n 0)\,|0\rangle\,d\mathfrak{r}_1\ldots d\mathfrak{r}_n \tag{1.5}$$

where the φ_a are the "wave" functions of the system. Therefore by these assumptions we describe the motion of undressed point particles being the condition for potential scattering.

For the following it is useful to introduce space like Fourier transforms by

$$\psi(\mathfrak{r},t) = (2\pi)^{-3/2}\int \exp\{-i\,\mathfrak{k}\,\mathfrak{r}\}\,\tilde{\psi}(\mathfrak{k},t)\,d\mathfrak{k} \tag{1.6}$$

leading to the commutation relation

$$[\tilde{\psi}(\mathfrak{k},t)\,\tilde{\psi}^+(\mathfrak{k}',t)]_- = \delta(\mathfrak{k}-\mathfrak{k}')\,\mathbf{1} \tag{1.7}$$

and to the field equation

$$\left(i\frac{\partial}{\partial t}-\mathfrak{k}^2\right)\tilde{\psi}(\mathfrak{k},t) - \int V(\mathfrak{k},\mathfrak{k}',\mathfrak{k}'',\mathfrak{k}''')\,\tilde{\psi}^+(\mathfrak{k}',t)\,\tilde{\psi}(\mathfrak{k}'',t)\,\tilde{\psi}(\mathfrak{k}''',t)\,d\mathfrak{k}'\,d\mathfrak{k}''\,d\mathfrak{k}'''=0 \tag{1.8}$$

with

$$V(\mathfrak{k},\mathfrak{k}',\mathfrak{k}'',\mathfrak{k}''') := \tilde{V}(\mathfrak{k}'''-\mathfrak{k})\,\delta(\mathfrak{k}+\mathfrak{k}'-\mathfrak{k}''-\mathfrak{k}''') \tag{1.9}$$

where $\tilde{V}$ is the Fouriertransformed potential.

For the functional treatment of this system we define $A_1 := \tilde{\psi}$ and $A_2 := \tilde{\psi}^+$. Then the generating functional reads

$$\mathfrak{T}_a(j) := \langle 0|\,T\exp\{i\int A_\alpha(\mathfrak{k},t)\,j_\alpha(\mathfrak{k},t)\,d\mathfrak{k}\,dt\}\,|a\rangle \tag{1.10}$$

with the corresponding functional equation

$$\left(\frac{\partial}{\partial t}B_{\alpha\beta}+\mathfrak{k}^2\,\delta_{\alpha\beta}\right)\frac{\delta}{\delta j_\beta(\mathfrak{k},t)}\mathfrak{T}_a(j) = \left[\delta_{\alpha\beta}\,C_{\gamma\delta}\int V(\mathfrak{k},\mathfrak{k}',\mathfrak{k}'',\mathfrak{k}''')\,\frac{\delta}{\delta j_\gamma(\mathfrak{k}',t)}\,\frac{\delta}{\delta j_\delta(\mathfrak{k}'',t)}\,\frac{\delta}{\delta j_\beta(\mathfrak{k}''',t)}\,d\mathfrak{k}'\,d\mathfrak{k}''\,d\mathfrak{k}'''+D_{\alpha\beta}\,j_\beta(\mathfrak{k},t)\right]\mathfrak{T}_a(j) \tag{1.11}$$

derived by the usual procedure [10] where the matrices are given by

$$B := \begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix},\quad C := \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},\quad D := \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{1.12}$$

For stationary state functionals the subsidiary conditions

$$\int j_\alpha(\mathfrak{k},t)\,\frac{\partial}{\partial t}\,\frac{\delta}{\delta j_\alpha(\mathfrak{k},t)}\,d\mathfrak{k}\,dt\,\mathfrak{T}_a(j) = i(E_a-E_0)\,\mathfrak{T}_a(j) \tag{1.13}$$

and

$$\int j_\alpha(\mathfrak{k},t)\,\mathfrak{k}\,\frac{\delta}{\delta j_\alpha(\mathfrak{k},t)}\,d\mathfrak{k}\,dt\,\mathfrak{T}_a(j) = -(\mathfrak{p}_a-\mathfrak{p}_0)\,\mathfrak{T}_a(j) \tag{1.14}$$

have to be satisfied additionally, where $E_a, \mathfrak{p}_a$ is the energy-momentum vector of the state $|a\rangle$. To avoid unnecessary indices we introduce the vector description by $\mathfrak{A} := (A_1\,A_2)$ and $\mathfrak{J} := (j_1\,j_2)$. With this abbreviation the state functionals can be written

$$\mathfrak{T}_a(\mathfrak{J}) := \langle 0|\,T\exp\{i\int \mathfrak{A}(\mathfrak{k},t)\,\mathfrak{J}(\mathfrak{k},t)\,d\mathfrak{k}\,dt\}\,|a\rangle. \tag{1.15}$$

As we shall see later, it is convenient to enclose the whole system in a large but finite box with periodic boundary conditions. Then the integration over the

[10] W. SCHULER and H. STUMPF, Z. Naturforsch. **22 a**, 1842 [1967].

$\mathfrak{k}$-range is changed into a summation over discrete $\mathfrak{k}$-values. Finally one may perform the limit to infinite volume i. e. the original description.

2. Asymptotic Free Fields

For the purpose of S-matrix construction we consider the asymptotic fields of ingoing and outgoing free particles. Denoting theirmatter field operator by $\chi(\mathfrak{r}, t)$ their Hamiltonian reads

$$H_{\mathrm{f}}(t) = - \int \nabla\chi^{+}(\mathfrak{r}, t)\, \nabla\chi(\mathfrak{r}, t)\, \mathrm{d}\mathfrak{r}\,, \quad (2.1)$$

where χ^{+} is the Hermitean conjugate of χ. It is $H(t) = H(0)$ and the commutation relation

$$[\chi(\mathfrak{r}, t)\, \chi^{+}(\mathfrak{r}', t)]_{-} = \delta(\mathfrak{r} - \mathfrak{r}')\, \mathbf{1} \quad (2.2)$$

is assumed, while all other commutators vanish. Then from (2.1) and (2.2) follows the field equation

$$\left(i\frac{\partial}{\partial t} + \Delta\right) \chi(\mathfrak{r}, t) = 0 \quad (2.3)$$

and it is assumed

$$\chi(\mathfrak{r}, t)\, |\, 0\rangle = 0\,. \quad (2.4)$$

Introducing space like Fouriertransforms by

$$\chi(\mathfrak{r}, t) = (2\,\pi)^{-3/2} \int \exp\{-i\,\mathfrak{k}\,\mathfrak{r}\}\, \tilde{\chi}\,(\mathfrak{k}, t)\, \mathrm{d}\mathfrak{k} \quad (2.5)$$

the commutation relation (2.2) is transformed into

$$[\tilde{\chi}\,(\mathfrak{k}, t)\, \tilde{\chi}^{+}(\mathfrak{k}', t)]_{-} = \delta(\mathfrak{k} - \mathfrak{k}')\, \mathbf{1} \quad (2.6)$$

with the Hamilton operator

$$\tilde{H}_{\mathrm{f}}(t) = \int \omega(\mathfrak{k})\, \tilde{\chi}^{+}(\mathfrak{k}, t)\, \tilde{\chi}\,(\mathfrak{k}, t)\, \mathrm{d}\mathfrak{k} \quad (2.7)$$

and the field equation

$$\left[i\frac{\partial}{\partial t} - \omega(\mathfrak{k})\right] \tilde{\chi}\,(\mathfrak{k}, t) = 0 \quad (2.8)$$

where $\omega(\mathfrak{k}) := \mathfrak{k}^2$.

Therefore by this transformation the system is represented by a set of uncoupled harmonic oscillators. Assuming the system to be enclosed in a large but finite box with periodic boundary conditions, the $\mathfrak{k}$-vectors become discrete and the eigenstates of the system are given by

$$|\,\mathfrak{N}\rangle \equiv |\, n_1 \dots n_N\rangle := \prod_{i=1}^{N} (n_i!)^{-1/2}\, \tilde{\chi}^{+}(\mathfrak{k}_i\, 0)^{n_i}\, |\, 0\rangle \quad (2.9)$$

where the index i characterizes the discrete set of $\mathfrak{k}$-values. For the functional description of this system we define $B_1 := \tilde{\chi}$ and $B_2 := \tilde{\chi}^{+}$. Then the generating functional reads

$$\mathfrak{T}_{\mathfrak{N}}^{\mathrm{f}}\,(j) := \langle 0\,|\, T\, \exp\{i \int B_{\alpha}(\mathfrak{k}, t)\, j_{\alpha}(\mathfrak{k}, t)\, \mathrm{d}\mathfrak{k}\, \mathrm{d}t\}\, |\, \mathfrak{N}\rangle \quad (2.10)$$

with the corresponding functional equation

$$\left(\frac{\partial}{\partial t} B_{\alpha\beta} + \omega(\mathfrak{k})\, \delta_{\alpha\beta}\right) \frac{\delta}{\delta j_{\beta}(\mathfrak{k}, t)}\, \mathfrak{T}_{\mathfrak{N}}^{\mathrm{f}}\,(j) = D_{\alpha\beta}\, j_{\beta}(\mathfrak{k}, t)\, \mathfrak{T}_{\mathfrak{N}}^{\mathrm{f}}\,(j) \quad (2.11)$$

and the stationarity conditions (1.13), (1.14) valid also for $\mathfrak{T}_{\mathfrak{N}}^{\mathrm{f}}$. For periodic boundary conditions the $\mathfrak{k}$-integrals degenerate to summations over discrete $\mathfrak{k}$-values. For the solution of (2.11) we observe (2.11) to be an uncoupled equation with regard to the $\mathfrak{k}$-values. Therefore for discrete $\mathfrak{k}$-values a solution is given by

$$\mathfrak{T}_{\mathfrak{N}}^{\mathrm{f}}\,(\mathfrak{J}) = \prod_{i} \mathfrak{T}_{n_i}(\mathfrak{J}_i) \quad (2.12)$$

where the $\mathfrak{T}_{n_i}$ are single harmonic oscillator functionals and we introduced vector variables

$$\mathfrak{J}_i := \{j_1(\mathfrak{k}_i\, t), j_2(\mathfrak{k}_i\, t)\}\,.$$

The general prescription for the formation of a functional scalar product [7, 8] can be applied to the special case of a harmonic oscillator [9]. Therefore the functional scalar product between two state-functionals $\mathfrak{T}_{\mathfrak{N}}^{\mathrm{f}}$ and $\mathfrak{T}_{\mathfrak{N}'}^{\mathrm{f}}$ can be defined according to Appendix I by

$$\langle \mathfrak{T}_{\mathfrak{N}}^{\mathrm{f}}\,(\mathfrak{J})\, \mathfrak{T}_{\mathfrak{N}'}^{\mathrm{f}}\,(\mathfrak{J})\, \rangle = \prod_{i} \langle \mathfrak{T}_{n_i}(\mathfrak{J}_i)\, \mathfrak{T}_{n'_i}(\mathfrak{J}_i)\, \rangle \quad (2.13)$$

i. e. the scalar product between two state functionals of the ensemble of harmonic oscillators is the product of the single harmonic oscillator scalar products. These products are

$$\langle \mathfrak{T}_{n_i}(\mathfrak{J}_i)\, \mathfrak{T}_{n'_i}(\mathfrak{J}_i)\, \rangle = \int \exp\{-\int \mathfrak{J}_i(t)\, \mathfrak{G}_i^{\mathrm{f}}(t, t')\, \mathfrak{J}_i(t')\, \mathrm{d}t\, \mathrm{d}t'\} \times \Gamma^{-1}(N_i)\, \mathfrak{T}_{n_i}^{\times}(\mathfrak{J}_i)\, \Gamma^{-1}(N_i)\, \mathfrak{T}_{n'_i}(\mathfrak{J}_i)\, \mathrm{d}\tau(\mathfrak{J}_i) \quad (2.14)$$

where the $\mathfrak{G}_i^{\mathrm{f}}$ are discussed in Appendix I. In [9] it is shown that the single harmonic oscillator functionals are orthonormalized with regard to the scalar product definition (2.14). Then by (2.13) follows

$$\langle \mathfrak{T}_{\mathfrak{N}}^{\mathrm{f}}\,(\mathfrak{J})\, \mathfrak{T}_{\mathfrak{N}'}^{\mathrm{f}}\,(\mathfrak{J})\, \rangle = \delta_{n_1 n'_1} \dots \delta_{n_N n'_N}\,. \quad (2.15)$$

In (2.9) resp. (2.12) the states are characterized by a quantum number representation. But as is known a particle number representation can be used equally well describing the distribution of the various quanta over the $\mathfrak{k}$-range of the set of oscillators. With $\sum_{i=1}^{N} n_i = m$ the number of quanta present in the set is m.

This leads to a representation by

$$\mathfrak{T}^{\mathrm{f}}_{n_1\dots n_N}(\mathfrak{J}) \equiv \mathfrak{T}^{\mathrm{f}}_{k_1\dots k_m}(\mathfrak{J}) \qquad (2.16)$$

with the orthonormality relations

$$\langle \mathfrak{T}^{\mathrm{f}}_{\mathfrak{K}}(\mathfrak{J})\, \mathfrak{T}^{\mathrm{f}}_{\mathfrak{K}'}(\mathfrak{J}) \rangle = \delta_{lm}\, \delta_{k_1 k'_1} \dots \delta_{k_m k'_m} \qquad (2.17)$$

where $\mathfrak{K}$ is the abbreviation of $\mathfrak{k}_1 \dots \mathfrak{k}_m$.

3. S-Matrix Construction

To construct the S-matrix we have to introduce some definitions first. Functionals like (1.10) or (2.10) are defined for Heisenberg states at the time $t=0$. More generally functionals can be defined by

$$\mathfrak{T}_a(\mathfrak{J}, \vartheta) := \langle 0 \,|\, T \exp\{i \int \mathfrak{A}(\mathfrak{k}, t)\, \mathfrak{J}(\mathfrak{k}, t)\, \mathrm{d}\mathfrak{k}\, \mathrm{d}t\} \,|\, a(\vartheta) \rangle \qquad (3.1)$$

where $|\, a(\vartheta) \rangle$ is a Schrödinger state at an arbitrary time ϑ. Defining the scalar product for Heisenberg functionals by

$$\langle \mathfrak{T}_a(\mathfrak{J}, 0)\, \mathfrak{T}_b(\mathfrak{J}, 0) \rangle := \int \exp\{-\mathfrak{J} \cdot \mathfrak{G}(0) \cdot \mathfrak{J}\}\, \Gamma^{-1} \mathfrak{T}^{\times}_a(\mathfrak{J}, 0)\, \Gamma^{-1} \mathfrak{T}_b(\mathfrak{J}, 0)\, \mathrm{d}\tau(\mathfrak{J}) \qquad (3.2)$$

with

$$\mathfrak{J} \cdot \mathfrak{G}(0) \cdot \mathfrak{J} := \int \mathfrak{J}(\mathfrak{k}, t)\, \mathfrak{G}(\mathfrak{k}, t, \mathfrak{k}', t')\, \mathfrak{J}(\mathfrak{k}', t')\, \mathrm{d}\mathfrak{k}\, \mathrm{d}t\, \mathrm{d}\mathfrak{k}'\, \mathrm{d}t' \qquad (3.3)$$

the more general scalar product expression for Schrödinger functionals is given by

$$\langle \mathfrak{T}_a(\mathfrak{J}, \vartheta)\, \mathfrak{T}_b(\mathfrak{J}, \vartheta) \rangle_\vartheta := \int \exp\{-\mathfrak{J} \cdot \mathfrak{G}(\vartheta) \cdot \mathfrak{J}\}\, \Gamma^{-1} \mathfrak{T}^{\times}_a(\mathfrak{J}, \vartheta)\, \Gamma^{-1} \mathfrak{T}_b(\mathfrak{J}, \vartheta)\, \mathrm{d}\tau(\mathfrak{J}) \qquad (3.4)$$

with

$$\mathfrak{G}(\vartheta) := \mathfrak{G}(\mathfrak{k}, t-\vartheta, \mathfrak{k}', t'-\vartheta). \qquad (3.5)$$

It can be shown that

$$\langle \mathfrak{T}_a(\mathfrak{J}, \vartheta)\, \mathfrak{T}_b(\mathfrak{J}, \vartheta) \rangle_\vartheta = \langle \mathfrak{T}_a(\mathfrak{J}, 0)\, \mathfrak{T}_b(\mathfrak{J}, 0) \rangle \qquad (3.6)$$

for arbitrary ϑ is valid. The proof is given in Appendix II.

Further the advanced and retarded functionals are required. They are defined by

$$\mathfrak{T}^{(\pm)}_a(\mathfrak{J}, \vartheta) := \langle 0 \,|\, T \exp\{i \int \mathfrak{A}(\mathfrak{k}, t)\, \mathfrak{J}(\mathfrak{k}, t)\, \mathrm{d}\mathfrak{k}\, \mathrm{d}t\} \,|\, a^{(\pm)}(\vartheta) \rangle \qquad (3.7)$$

where $|\, a^{(\pm)}(\vartheta) \rangle$ are the corresponding advanced resp. retarded Schrödinger states. Then we have according to (3.6)

$$\langle \mathfrak{T}^{(+)}_{\mathfrak{K}}(\mathfrak{J}, 0)\, \mathfrak{T}^{(-)}_{\mathfrak{K}'}(\mathfrak{J}, 0) \rangle = \langle \mathfrak{T}^{(+)}_{\mathfrak{K}}(\mathfrak{J}, \vartheta)\, \mathfrak{T}^{(-)}_{\mathfrak{K}'}(\mathfrak{J}, \vartheta) \rangle_\vartheta \qquad (3.8)$$

where $\mathfrak{K}$ and $\mathfrak{K}'$ are the sets of quantum numbers of the ingoing resp. outgoing free particles which characterize the advanced and retarded states resp. functionals completly.

By direct evaluation we obtain according to [7]

$$\langle \mathfrak{T}^{(+)}_{\mathfrak{K}}(\mathfrak{J}, \vartheta)\, \mathfrak{T}^{(-)}_{\mathfrak{K}'}(\mathfrak{J}, \vartheta) \rangle_\vartheta = \sum_{k,l,\varDelta} \frac{c^{kl}_{r}}{(k!\, l!)^2} \int \tau^{(+)}_{\mathfrak{K}\vartheta}(\mathfrak{k}_1 t_1 \dots \mathfrak{k}_k t_k)\, \tau^{(-)}_{\mathfrak{K}'\vartheta}(\mathfrak{k}_1' t_1' \dots \mathfrak{k}_l' t_l')$$
$$\times \mathfrak{R}^\vartheta(\mathfrak{k}_1 t_1 \mathfrak{k}_1' t_1') \dots \mathfrak{R}^\vartheta(\mathfrak{k}_{l-1}' t_{l-1}' \mathfrak{k}_l' t_l')\, \mathrm{d}\mathfrak{k}_1\, \mathrm{d}t_1 \dots \mathrm{d}\mathfrak{k}_l'\, \mathrm{d}t_l' \qquad (3.9)$$

where the generalized time ordered products for Schrödigner states are labelled with ϑ and

$$\mathfrak{R}^\vartheta(\mathfrak{k}, t, \mathfrak{k}', t') := \mathfrak{G}^{-1}(\mathfrak{k}, t-\vartheta, \mathfrak{k}', t'-\vartheta). \qquad (3.10)$$

Although we calculate the scalar product for interacting field functionals, we assume now $\mathfrak{G} \equiv \mathfrak{G}^{\mathrm{f}}$ i. e. we use the weightfunction of free fields for the evaluation of the functional scalar products (3.10). This assumption shall be justified later. Using $\mathfrak{G}^{\mathrm{f}}$ one observes that the corresponding $\mathfrak{R}_{\mathrm{f}}^{\vartheta}$ is essentially unequal zero only in a small region around the points $t=\vartheta$ and $t'=\vartheta$. This is discussed in detail in Appendix I. As a consequence of this property the integration in (3.9) is confined essentially to a small region around the points $t_1 = \dots = t_k = t_1' \dots = t_l' = \vartheta$. Writing for $t = \vartheta + \varepsilon$

$$\tau^{(\pm)}_{\mathfrak{K},\vartheta}(\mathfrak{k}_1 t_1 \dots \mathfrak{k}_k t_k) = \tau^{(\pm)}_{\mathfrak{K},\vartheta}(\mathfrak{k}_1, \vartheta + \varepsilon_1, \dots \mathfrak{k}_k, \vartheta + \varepsilon_k) \qquad (3.11)$$

we may introduce a spectral decomposition with

$$\tau^{(\pm)}_{\mathfrak{K}\vartheta}(\mathfrak{k}_1 t_1 \dots \mathfrak{k}_k t_k) = \mathrm{P} \sum_{\lambda_1 \dots \lambda_k} \sum_{m_1 \dots m_k} \langle 0 \,|\, \mathfrak{A}(\mathfrak{k}_{\lambda_1}, \vartheta + \varepsilon_{\lambda_1}) \,|\, m_1 \rangle \dots \langle m_{-1} \,|\, \mathfrak{A}(\mathfrak{k}_{\lambda k}, \vartheta + \varepsilon_{\lambda k}) \,|\, a^{(\pm)}_{\mathfrak{K}}(\vartheta) \rangle\, \Theta(\varepsilon_{\lambda_1} \dots \varepsilon_{\lambda k}) \qquad (3.12)$$

where the $|m_\alpha\rangle$ are a complete system of intermediate states. Further we have

$$\lim_{\vartheta\to\infty} |a_{\mathfrak{K}'}^{(-)}(\vartheta)\rangle = |\mathfrak{K}'\rangle \tag{3.13}$$

and

$$\lim_{\vartheta\to\infty} |a_{\mathfrak{K}}^{(+)}(\vartheta)\rangle = \sum_{\mathfrak{K}''} S_{\mathfrak{K}\mathfrak{K}''} |\mathfrak{K}''\rangle \tag{3.14}$$

i. e. the ingoing and outgoing states are decomposed into free particle states, leading in this way by (3.14) to the definition of the S-matrix. Now we assume the L.S.Z. asymptotic condition to be valid [11]. For simplicity we use local field operators instead of quasilocal ones. Then this condition reads

$$\lim_{\vartheta\to\infty} \langle m_\alpha | \mathfrak{A}(\mathfrak{k}, \vartheta+\varepsilon) | m_\beta\rangle = \lim_{\vartheta\to\infty} \langle m_\alpha | \mathfrak{B}(\mathfrak{k}, \vartheta+\varepsilon) | m_\beta\rangle \tag{3.15}$$

for any finite ε and we obtain by substitution of (3.13), (3.14), (3.15) into (3.12)

$$\begin{aligned}\lim_{\vartheta\to\infty} \tau_{\mathfrak{K},\vartheta}^{(+)}(\mathfrak{k}_1 t_1 \ldots \mathfrak{k}_k t_k) &= \lim_{\vartheta\to\infty} \sum_{\mathfrak{K}''} S_{\mathfrak{K}\mathfrak{K}''} \, \mathrm{P} \sum_{\lambda_1\ldots\lambda_k} \langle 0 | \mathfrak{B}(\mathfrak{k}_{\lambda_1}, \vartheta+\varepsilon_{\lambda_1}) \ldots \mathfrak{B}(\mathfrak{k}_{\lambda_k}, \vartheta+\varepsilon_{\lambda_k}) | \mathfrak{K}''\rangle \, \Theta(\varepsilon_{\lambda_1} \ldots \varepsilon_{\lambda_k}) \\ &= \lim_{\vartheta\to\infty} \sum_{\mathfrak{K}''} S_{\mathfrak{K}\mathfrak{K}''} \, \tau_{\mathfrak{K}''\vartheta}^{\mathrm{f}}(\mathfrak{k}_1, \vartheta+\varepsilon_1, \ldots, \mathfrak{k}_k, \vartheta+\varepsilon_k)\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}\lim_{\vartheta\to\infty} \tau_{\mathfrak{K}'\vartheta}^{(-)}(\mathfrak{k}_1' t_1' \ldots \mathfrak{k}_l' t_l') &= \lim_{\vartheta\to\infty} \mathrm{P} \sum_{\mu_1\ldots\mu_l} \langle 0 | \mathfrak{B}(\mathfrak{k}_{\mu_1}', \vartheta+\varepsilon_{\mu_1}') \ldots \mathfrak{B}(\mathfrak{k}_{\mu l}', \vartheta+\varepsilon_{\mu l}') | \mathfrak{K}'\rangle \, \Theta(\varepsilon_{\mu_1}' \ldots \varepsilon_{\mu l}') \\ &= \lim_{\vartheta\to\infty} \tau_{\mathfrak{K}'\vartheta}^{\mathrm{f}}(\mathfrak{k}_1', \vartheta+\varepsilon_1' \ldots \mathfrak{k}_l', \vartheta+\varepsilon_l').\end{aligned} \tag{3.17}$$

Because in (3.9) the integration runs only over a small region around ϑ for each t and t' variable, we are allowed to substitute the functions in (3.9) by their limes expressions (3.16) and (3.17). But this leads exactly to the relation

$$\begin{aligned}&\lim_{\vartheta\to\infty} \langle \mathfrak{T}_{\mathfrak{K}}^{(+)}(\mathfrak{J}, \vartheta) \, \mathfrak{T}_{\mathfrak{K}'}^{(-)}(\mathfrak{J}, \vartheta) \rangle_{\vartheta, \mathrm{f}} \\ &\quad = \sum_{\mathfrak{K}''} S_{\mathfrak{K}\mathfrak{K}''} \lim_{\vartheta\to\infty} \langle \mathfrak{T}_{\mathfrak{K}''}^{\mathrm{f}}(\mathfrak{J}, \vartheta) \, \mathfrak{T}_{\mathfrak{K}'}^{\mathrm{f}}(\mathfrak{J}, \vartheta) \rangle_{\vartheta}\end{aligned} \tag{3.18}$$

where the index f on the left side of (3.18) indicates that this product is formed by $\mathfrak{G}^{\mathrm{f}}$. Now for free fields the scalar product is given by (2.17). Inserting this into (3.18) and observing (3.8) we obtain finally

$$S_{\mathfrak{K}\mathfrak{K}'} = \langle \mathfrak{T}_{\mathfrak{K}}^{(+)}(\mathfrak{J}, 0) \, \mathfrak{T}_{\mathfrak{K}'}^{(-)}(\mathfrak{J}, 0) \rangle_{\mathrm{f}}. \tag{3.19}$$

Therefore we see that we obtain the S-matrix by funcional scalar-products using the weight function for free fields, which justifies our assumption about $\mathfrak{G}$ made before. It is essentially this feature which enables ourself to perform scattering calculations by means of functionals not only theoretically but also numerically because $\mathfrak{G}^{\mathrm{f}}$ is explicitly known. Another question is the expliict calculation of scattering functionals themselves from given initial conditions of outgoing resp. ingoing particle states to be scattered. This question shall be discussed in the next section.

4. Calculation of Scattering Functionals

In the preceding section we derived the functional representation of the S-matrix using the exact scattering functionals. But to complete functional quantum theory of scattering also a calculational method for the explicit construction of scattering functionals is required. This is provided by the method of least squares applied already successfully to Bethe-Salpeter boundstate and scattering problems [12, 13] and proposed for the functional equation in [7]. To apply this method to functional scattering problems, one has to know the appropriate boundary or initial conditions which have to be satisfied by scattering functionals themselves. To derive them we consider first the scattering states in physical Hilbert space. It is sufficient to consider only two particle scattering states. All other cases can be treated on the same pattern. The two particle states read

$$\begin{aligned}|a_{\mathfrak{K}}^{(\pm)}(0)\rangle := \int \varphi_{\mathfrak{K}}^{(\pm)}(\mathfrak{r}_1 \mathfrak{r}_2) \, \psi^+(\mathfrak{r}_1 0) \, \psi^+(\mathfrak{r}_2 0) \\ \cdot |0\rangle \, d\mathfrak{r}_1 \, d\mathfrak{r}_2.\end{aligned} \tag{4.1}$$

According to Schrödinger theory $\varphi_{\mathfrak{K}}^{(\pm)}$ with $\mathfrak{K} \equiv (\mathfrak{k}_1 \mathfrak{k}_2)$ can be calculated and has the general form

$$\begin{aligned}\varphi_{\mathfrak{k}_1\mathfrak{k}_2}^{(\pm)}(\mathfrak{r}_1 \mathfrak{r}_2) = \exp\{-i(\mathfrak{r}_1 \mathfrak{k}_1 + \mathfrak{r}_2 \mathfrak{k}_2)\} \\ + \Phi_{\mathfrak{k}_1\mathfrak{k}_2}^{(\pm)}(\mathfrak{r}_1 \mathfrak{r}_2)\end{aligned} \tag{4.2}$$

[11] R. Hagedorn, Fortschr. Phys., 5. Sonderband, 1963.
[12] K. Ladanyi, Nuovo Cim. **56 A,** 173 [1968].
[13] K. Ladanyi, Nuovo Cim. **61** A, 173 [1969].

where the first term describes the outgoing resp. ingoing particles while the second term is due to their mutual interaction. Substituting this into the definition of the scattering functionals (3.7) for $\vartheta=0$ we obtain

$$\mathfrak{T}^{(\pm)}_{\mathfrak{k}_1\mathfrak{k}_2}(\mathfrak{J},0)=\langle 0\,|\,\exp\{i\int\mathfrak{A}(\mathfrak{k},t)\,\mathfrak{J}(\mathfrak{k},t)\,\mathrm{d}\mathfrak{k}\,\mathrm{d}t\}\,A_2(\mathfrak{k}_1\,0)\,A_2(\mathfrak{k}_2\,0)\,|\,0\rangle+\mathfrak{D}^{(\pm)}_{\mathfrak{k}_1\mathfrak{k}_1}(\mathfrak{J},0) \quad (4.3)$$

where $\mathfrak{D}$ contains that part of the functional coming from Φ. The further evaluation of (4.3) can be done by observing

$$\mathfrak{A}(\mathfrak{k},t)=e^{-iHt}\,\mathfrak{A}(\mathfrak{k},0)\,e^{iHt} \quad (4.4)$$

and $$\mathfrak{B}(\mathfrak{k},t)=e^{-iH_{\mathrm{f}}t}\,\mathfrak{B}(\mathfrak{k},0)\,e^{iH_{\mathrm{f}}t}\,. \quad (4.5)$$

Now for $t=0$ the free field operators $\mathfrak{B}$ and the interacting field operators $\mathfrak{A}$ are equivalent because they obey the same commutation relation and can be defined in the same Hilbert space. Therefore it is $\mathfrak{A}(\mathfrak{k},0)=\mathfrak{B}(\mathfrak{k},0)$ and we may write the Hamiltonian (1.1)

$$H=H_{\mathrm{f}}+V \quad (4.6)$$

where V is the interaction operator. From this follows

$$\mathfrak{A}(\mathfrak{k},t)=e^{-iVt}\,\mathfrak{B}(\mathfrak{k},t)\,e^{iVt} \quad (4.7)$$

in the interaction representation and by evaluating the exponentials we have

$$\mathfrak{A}(\mathfrak{k},t)=\mathfrak{B}(\mathfrak{k},t)+\mathfrak{R}(\mathfrak{k},t) \quad (4.8)$$

where $\mathfrak{R}(\mathfrak{k},t)$ is defined by the remaining power series terms of (4.7). Its explicit form is not required. Replacing in (4.3) $A_2(\mathfrak{k}_i\,0)$ by $B_2(\mathfrak{k}_i\,0)$ and substituting (4.8) an expansion of the exponential shows that (4.3) can be written

$$\mathfrak{T}^{(\pm)}_{\mathfrak{k}_1\mathfrak{k}_2}(\mathfrak{J},0)=\mathfrak{T}^{\mathrm{f}}_{\mathfrak{k}_1\mathfrak{k}_2}(\mathfrak{J},0)+\mathfrak{D}^{(\pm)}_{\mathfrak{k}_1\mathfrak{k}_2}(\mathfrak{J},0) \quad (4.9)$$

where ϑ contains ϑ' and all other terms with at least one R operator. So we see, that also the scattering functionals can be separated into one part describing the free in- or outgoing particles and another resulting of their mutual interaction. Applying now the variational principle we symmetrize the operator equation according to [10, 14], write it in the symbolic notation

$$\boldsymbol{O}\left(\mathfrak{J},\frac{\delta}{\delta\mathfrak{J}}\right)\mathfrak{T}^{(\pm)}_{\mathfrak{K}}(\mathfrak{J},0)=0 \quad (4.10)$$

and require

$$\left\|\boldsymbol{O}\left(\mathfrak{J},\frac{\delta}{\delta\mathfrak{J}}\right)\mathfrak{T}^{(\pm)}_{\mathfrak{K}}(\mathfrak{J},0)\right\|^{\mathrm{f}}=\min\,. \quad (4.11)$$

For trial functionals we use the approximate functionals

$$\mathfrak{T}^{(\pm)}_{N\mathfrak{K}}(\mathfrak{J},0)=\mathfrak{T}^{\mathrm{f}}_{N\mathfrak{K}}(\mathfrak{J},0)+\mathfrak{D}^{(\pm)}_{N\mathfrak{K}}(\mathfrak{J},0) \quad (4.12)$$

where the index N means truncation in the usual way [7, 10, 14]. But now, contrary to the eigenvalue calculation, we are only free to vary the interaction functional $\mathfrak{D}^{(\pm)}_{N\mathfrak{K}}$, while the functional $\mathfrak{T}^{\mathrm{f}}_{N\mathfrak{K}}$ is a fixed boundary condition which forces the functional $\mathfrak{T}^{(\pm)}_{N\mathfrak{K}}$ to become a scattering functional describing in an outgoing particles with momentum $\mathfrak{k}_1$, $\mathfrak{k}_2$. Finally we mention that in the Heisenberg representation the advanced and retarded functions and functionals are calculated by the same equations. Their distinction results only from their derivation in the Schrödinger picture but not from their calculation.

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Appendix I

In this appendix we shall discuss the functional treatment of single harmonic oscillators and ensembles of them. We discuss first a single harmonic oscillator. Usually for the functional representation the dynamical variables $q=\psi_1$ and $p=\psi_2$ are used, and the functional is defined by

$$\mathfrak{T}^{\mathrm{f}}_{n}(h):=\langle 0\,|\,T\exp\{i\int\psi_\alpha(t)\,h_\alpha(t)\,\mathrm{d}t\}\,|\,n\rangle \quad (\mathrm{I.1})$$

where $|\,n\rangle$ is an eigenstate of the harmonic oscillator. But in the preceding section we used creation- and destruction operators a and a^+ with the definition $a=B_1$ and $a^+=B_2$, and the corresponding functional was defined by

$$\mathfrak{T}^{\mathrm{f}}_{n}(j):=\langle 0\,|\,T\exp\{i\int B_\alpha(t)\,j_\alpha(t)\,\mathrm{d}t\}\,|\,n\rangle\,. \quad (\mathrm{I.2})$$

The relation between both definitions is achieved by means of the canonical transformation

$$B_\alpha(t)=K_{\alpha\beta}\,\psi_\beta(t) \quad (\mathrm{I.3})$$

with $$K_{\alpha\beta}=\frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{\omega} & i/\sqrt{\omega}\\ \sqrt{\omega} & -i/\sqrt{\omega}\end{pmatrix} \quad (\mathrm{I.4})$$

[14] W. SCHULER and H. STUMPF, Z. Naturforsch. 23 a, 902 [1968].

where ω is the frequency of the harmonic oscillator. Defining then

$$h_\alpha(t) = j_\beta(t)\, K_{\beta\alpha} \qquad (I.5)$$

we have

$$\mathfrak{T}_n^{\mathrm{f}}(h) = \mathfrak{T}_n^{\mathrm{f}}(j) \qquad (I.6)$$

and all results for the ψ-representation can be transformed into the B-representation. In the ψ-representation the scalar product is defined by

$$\langle \mathfrak{T}_{n'}^{\mathrm{f}}(h)\, \mathfrak{T}_n^{\mathrm{f}}(h) \rangle = \int \cdot \exp\{-\int h_\alpha(t)\, H_{\alpha\beta}^{\mathrm{f}}(t,t')\, h_\beta(t')\, \mathrm{d}t\, \mathrm{d}t'\} \times \Gamma^{-1} \mathfrak{T}_{n'}^{\mathrm{f}\times}(h)\, \Gamma^{-1} \mathfrak{T}_n^{\mathrm{f}}(h)\, \mathrm{d}\tau(h) \qquad (I.7)$$

where the weight function $H_{\alpha\beta}^{\mathrm{f}}(t,t')$ has to be real and symmetric in the variables t and t'. Moreover it has to generate a positive symmetric form for the real functions $h_\alpha(t)$ out of the space S (the infinitely often differentiable functions which decrease faster at infinity than any power of their argument together with all their derivatives).

For the special choice

$$H_{\alpha\beta}^{\mathrm{f}}(t,t') \equiv \delta(t-t')\, g(t)\, D_{\alpha\beta} \qquad (I.8)$$

with $g(t)$ real, positive and $\int g^{-1}(t)\, \mathrm{d}t < \infty$ and

$$D = \begin{pmatrix} 1/\omega & 0 \\ 0 & \omega \end{pmatrix}$$

the function $g(t)$ can be calculated to give [9]

$$\langle \mathfrak{T}_n^{\mathrm{f}}(h)\, \mathfrak{T}_{n'}^{\mathrm{f}}(h) \rangle = \delta_{nn'}. \qquad (I.9)$$

Transforming the B_α we recognize that $\mathrm{d}\tau(h) \equiv \mathrm{d}\tau(j)$ and the functional scalar product (I.7) goes over into

$$\langle \mathfrak{T}_{n'}^{\mathrm{f}}(j)\, \mathfrak{T}_n^{\mathrm{f}}(j) \rangle = \int \exp\{-\int j_\alpha(t)\, G_{\alpha\beta}^{\mathrm{f}}(t,t')\, j_\beta^{\times}(t')\, \mathrm{d}t\, \mathrm{d}t'\} \times \Gamma^{-1} \mathfrak{T}_{n'}^{\mathrm{f}\times}(j)\, \Gamma^{-1} \mathfrak{T}_n^{\mathrm{f}}(j)\, \mathrm{d}\tau(j) \qquad (I.10)$$

with

$$G_{\alpha'\beta'}^{\mathrm{f}}(t,t') = K_{\alpha'\alpha}\, H_{\alpha\beta}^{\mathrm{f}}(t,t')\, K_{\beta\beta'}^{+} \qquad (I.11)$$

which becomes using (I.8)

$$G_{\alpha\beta}^{\mathrm{f}}(t,t') = \delta(t-t')\, \delta_{\alpha\beta}\, g(t). \qquad (I.11a)$$

Of course (I.9) is valid also in the transformed representation

$$\langle \mathfrak{T}_{n'}^{\mathrm{f}}(j)\, \mathfrak{T}_n^{\mathrm{f}}(j) \rangle = \delta_{n'n}. \qquad (I.12)$$

According to [7] the reciprocal $\mathfrak{R}$ introduced in (3.10) are defined by

$$\int R_{\alpha\beta}^{\mathrm{f}}(t,t'')\, G_{\beta\delta}^{\mathrm{f}}(t''\, t')\, \mathrm{d}t'' = \delta_{\alpha\delta}\, \delta(t-t') \qquad (I.13)$$

and from this follows that R has to be

$$R_{\alpha\beta}^{\mathrm{f}}(t,t') = \delta_{\alpha\beta}\, \delta(t-t')\, g^{-1}(t). \qquad (I.14)$$

The translated reciprocals (3.10) are given then by

$$R_{\alpha\beta}^{\mathrm{f}\vartheta}(t,t') = R_{\alpha\beta}^{\mathrm{f}}(t-\vartheta, t'-\vartheta) \qquad (I.15)$$

satisfying the relation

$$\int R_{\alpha\beta}^{\mathrm{f}\vartheta}(t,t'')\, G_{\beta\lambda}^{\mathrm{f}\vartheta}(t'',t')\, \mathrm{d}t'' = \delta_{\alpha\lambda}\, \delta(t-t') \qquad (I.16)$$

Now according to (I.8) and [9] $g^{-1}(t)$ has to be a rapidly decreasing function. Therefore approximately we use only the first term in an expansion of $g^{-1}(t)$ in harmonic oscillator functions $f_n(t)$

$$R_{\alpha\beta}^{\mathrm{f},\vartheta}(t,t') \approx \delta_{\alpha\beta}\, \delta(t-t')\, g_0^{-1}\, f_0(t-\vartheta) \qquad (I.17)$$

where $f_0(t)$ is the ground-state function of the harmonic oscillator. This function is centered around the origin and therefore $f_0(t-\vartheta)$ is a Gaussian weight function centered around $t=\vartheta$.

Considering an ensemble of uncoupled harmonic oscillators with the creation- and destruction operators $B_1(k)$ and $B_2(k)$ according to (2.12) the ensemble functional is defined by

$$\mathfrak{T}_{\mathfrak{N}}^{\mathrm{f}}(j) := \langle 0 | T \exp\{i \sum_\lambda \int B_\alpha(\mathfrak{k}_\lambda t)\, j(\mathfrak{k}_\lambda t)\, \mathrm{d}t\} | \mathfrak{N} \rangle \qquad (I.18)$$

where $|\mathfrak{N}\rangle$ are the ensemble states defined in (2.9). We show first by an argument independent of section 2 that (I.18) decomposes into the product of single harmonic oscillator functionals. Applying the Wick rule (I.18) goes over into

$$\mathfrak{T}_{\mathfrak{N}}^{\mathrm{f}}(j) = \exp\{\sum_{\lambda\lambda'} \int j_\alpha(\mathfrak{k}_\lambda t)\, F_{\alpha\beta}(\mathfrak{k}_\lambda t, \mathfrak{k}_{\lambda'} t')\, j_\beta(\mathfrak{k}_{\lambda'} t')\, \mathrm{d}t\, \mathrm{d}t'\}\, \Phi_{\mathfrak{N}}^{\mathrm{f}}(j) \qquad (I.19)$$

with

$$\Phi_{\mathfrak{N}}^{\mathrm{j}}(j) := \langle 0 | N \exp\{i \sum_\lambda \int B_\alpha(\mathfrak{k}_\lambda t)\, j_\alpha(\mathfrak{k}_\lambda t)\, \mathrm{d}t\} | \mathfrak{N} \rangle. \qquad (I.20)$$

Observing now for uncoupled oscillators the relations

$$F_{\alpha\beta}(\mathfrak{k}_\lambda, t, \mathfrak{k}_{\lambda'} t') = \delta_{\mathfrak{k}_\lambda \mathfrak{k}_{\lambda'}}\, F_{\alpha\beta}(t-t', \mathfrak{k}_\lambda) \quad \text{and} \quad |\mathfrak{N}\rangle = |n_1\rangle \times \ldots \times |n_N\rangle \qquad (I.21, 22)$$

from (I.17) follows immediately

$$\mathfrak{T}_{\mathfrak{N}}^{\mathrm{f}}(j) = \prod_i \mathfrak{T}_{n_i}^{\mathrm{f}}(j_i). \qquad (I.23)$$

This decomposition can be used for the evaluation of scalar products. In general the scalar product for the ensemble functional should be defined by

$$\langle \mathfrak{T}^{\mathfrak{f}}_{\mathfrak{N}'}(j)\, \mathfrak{T}^{\mathfrak{f}}_{\mathfrak{N}}(j) \rangle = \int \exp\{-\sum_{\lambda,\lambda'} \int j_\alpha(\mathfrak{f}_\lambda t)\, G^{\mathfrak{f}}_{\alpha\beta}(\mathfrak{f}_\lambda t, \mathfrak{f}_{\lambda'} t')\, j_\beta(\mathfrak{f}_{\lambda'} t')\, dt\, dt'\} \times \prod_{\lambda=1}^{N} \Gamma^{-1}(N_\lambda)\, \mathfrak{T}^{\mathfrak{f}\times}_{\mathfrak{N}'}(j) \prod_{\lambda'=1}^{N} \Gamma^{-1}(N_{\lambda'})\, \mathfrak{T}^{\mathfrak{f}}_{\mathfrak{N}'}(j) \prod_{\mu=1}^{N} d\tau(j_\mu) \quad (I.24)$$

with $j_\mu(t) := j(\mathfrak{f}_\mu t)$ for discrete $\mathfrak{f}_\mu$.

But now the decomposition (I.23) suggests to choose

$$G^{\mathfrak{f}}_{\alpha\beta}(\mathfrak{f}_\lambda t, \mathfrak{f}_{\lambda'} t') = \delta_{\mathfrak{f}_\lambda \mathfrak{f}_{\lambda'}}\, \delta(t-t')\, g(t)\, \delta_{\alpha\beta} \quad (I.25)$$

which results into

$$\langle \mathfrak{T}^{\mathfrak{f}}_{\mathfrak{N}'}(j)\, \mathfrak{T}^{\mathfrak{f}}_{\mathfrak{N}}(j) \rangle = \prod_{\lambda=1}^{N} \langle \mathfrak{T}^{\mathfrak{f}}_{n\lambda'}(j_\lambda)\, \mathfrak{T}^{\mathfrak{f}}_{n\lambda}(j_\lambda) \rangle \quad (I.26)$$

Then R^ϑ is given by

$$R^\vartheta_{\alpha\beta}(\mathfrak{f}_\lambda t, \mathfrak{f}_{\lambda'} t') = \delta_{\mathfrak{f}_\lambda \mathfrak{f}_{\lambda'}}\, \delta(t-t')\, g^{-1}(t-\vartheta)\, \delta_{\alpha\beta} \quad (I.27)$$

and the same arguments are valid as for (I.15). For the limit of continuous $\mathfrak{f}$ the definition of (I.24) has to be proven to make sense. In this case the scalar product shall be defined by the limiting expression of (I.26) which exists because of the orthonormality condition (I.12) of the different states.

Appendix II

To prove the equality (3.6) i.e. to prove the invariance of the functional scalar product for different times ϑ we consider first for convenience the special case of the norm expression $\| \mathfrak{T}(\mathfrak{J}, \vartheta) \|_\vartheta^2$. Because in the following proof the dependence of $\mathfrak{J}$ and $\mathfrak{A}$ on $\mathfrak{f}$ does not play any role, we omit these variables and write in a selfexplanatory way

$$\| \mathfrak{T}(\mathfrak{J}, \vartheta) \|_\vartheta^2 = \int \exp\{-\int \mathfrak{J}(t)\, \mathfrak{G}^\vartheta(t,t')\, \mathfrak{J}(t')\, dt\, dt'\} \times |\Gamma^{-1}(N)\, \mathfrak{T}(\mathfrak{J}, \vartheta)|^2\, d\tau(\mathfrak{J}). \quad (II.1)$$

Substituting the definition of the weight function and the functional we may write

$$\| \mathfrak{T}(\mathfrak{J}, \vartheta) \|_\vartheta^2 = \int \exp\{-\int \mathfrak{J}(t)\, \mathfrak{G}(t-\vartheta, t'-\vartheta)\, \mathfrak{J}(t')\, dt\, dt'\} \times |\Gamma^{-1}(N) \langle 0 | T \quad (II.2)$$
$$\times \exp\{i \int \mathfrak{A}(t)\, \mathfrak{J}(t)\, dt\} | a(\vartheta) \rangle |^2\, d\tau(\mathfrak{J}).$$

Substituting now the new variables $\xi = t - \vartheta$ for all t integrations we obtain

$$\| \mathfrak{T}(\mathfrak{J}, \vartheta) \|_\vartheta^2 = \int \exp\{-\int \mathfrak{J}(\xi+\vartheta)\, \mathfrak{G}(\xi\, \xi')\, \mathfrak{J}(\xi'+\vartheta)\, d\xi\, d\xi'\} \times |\Gamma^{-1}(N) \langle 0 | T \quad (II.3)$$
$$\times \exp\{i \int \mathfrak{A}(t)\, (\xi+\vartheta)\, \mathfrak{J}(\xi+\vartheta)\, d\xi\} | a(\vartheta) \rangle |^2\, d\tau(\mathfrak{J}).$$

Now we replace the source functions $\mathfrak{J}(\xi+\vartheta)$ by

$$\mathfrak{H}(\xi) := \mathfrak{J}(\xi+\vartheta) \quad (II.4)$$

being a set of new source functions. Introducing for an intermediate step a rigged Hilbert space representation, we have

$$\mathfrak{J}(\xi) = \sum \varrho_k\, f_k(\xi) \quad (II.5)$$

and

$$\mathfrak{H}(\xi) = \sum \varrho_k\, f_k(\xi+\vartheta). \quad (II.6)$$

Expanding the set $f_k(\xi+\vartheta)$ by the set $f_k(\xi)$ we obtain a unitary transfromation

$$f_k(\xi+\vartheta) = \sum_l u_{kl}(\vartheta)\, f_l(\xi) \quad (II.7)$$

resulting in

$$\mathfrak{H}(\xi) = \sum_{k,l} \varrho_k\, u_{kl}(\vartheta)\, f_l(\xi) = \sum_l \varrho_l'\, f_l(\xi). \quad (II.8)$$

Now because of the unitarity it is $d\tau(\varrho) = d\tau(\varrho')$ and turning back to the original representation of $\mathfrak{J}$ and $\mathfrak{H}$ we have $d\tau(\mathfrak{J}) = d\tau(\mathfrak{H})$ therefore (II.3) can be written

$$\| \mathfrak{T}(\mathfrak{J}, \vartheta) \|_\vartheta^2 = \int \exp\{-\int \mathfrak{H}(\xi)\, \mathfrak{G}(\xi, \xi')\, \mathfrak{H}(\xi')\, d\xi\, d\xi'\} \times |\Gamma^{-1}(N) \langle 0 | T \quad (II.9)$$
$$\times \exp\{i \int \mathfrak{A}(\xi+\vartheta)\, \mathfrak{H}(\xi)\, d\xi\} | a(\vartheta) \rangle |^2\, d\tau(\mathfrak{H}).$$

Observing finally

$$\mathfrak{A}(\xi+\vartheta) = e^{-iH\vartheta}\, \mathfrak{A}(\xi)\, e^{iH\vartheta} \quad (II.10)$$

with

$$\langle 0 | e^{-iH\vartheta} = \langle 0 | \quad (II.11)$$

and

$$e^{iH\vartheta} | a(\vartheta) \rangle = | a(0) \rangle \quad (II.12)$$

we obtain from (II.9)

$$\| \mathfrak{T}(\mathfrak{J}, \vartheta) \|_\vartheta^2 = \int \exp\{-\int \mathfrak{H}(\xi)\, \mathfrak{G}(\xi, \xi')\, \mathfrak{H}(\xi')\, d\xi\, d\xi'\} \times |\Gamma^{-1}(N) \langle 0 | T \quad (II.13)$$
$$\times \exp\{i \int \mathfrak{A}(\xi)\, \mathfrak{H}(\xi)\, d\xi\} | a(0) \rangle |^2\, d\tau(\mathfrak{H})$$

and from this follows

$$\| \mathfrak{T}(\mathfrak{J}, \vartheta) \|_\vartheta^2 = \| \mathfrak{T}(\mathfrak{H}, 0) \|_0^2 = \| \mathfrak{T}(\mathfrak{J}, 0) \|_0^2 \quad (II.14)$$

because the evaluation of the integral does not depend upon the symbol used for the sources. As one recognizes easily the proof does not depend on the use of a norm expression. Rather one could have used a scalar product equally well. Therefore (3.5) has been proven.